Bak-Sneppen model near zero dimension

S. N. Dorogovtsev,[1,2,*] J. F. F. Mendes,[1,†] and Yu. G. Pogorelov[1,‡]
[1]*Departamento de Física and Centro de Física do Porto, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal*
[2]*A. F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia*

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We consider the Bak-Sneppen model near zero dimension, where the avalanche exponent τ is close to 1 and the exponents μ and σ are close to 0. We demonstrate that $\tau-1=\mu-\sigma=\exp\{-\mu^{-1}-\gamma+\cdots\}$ in this limit, where γ is Euler's constant. The avalanche hierarchy equation is rewritten in a form that makes it possible to find the relation between the critical exponents σ and μ with high accuracy. We obtain precise values of the critical exponents for the one and two-dimensional Bak-Sneppen model and for the one-dimensional anisotropic Bak-Sneppen model.

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Perhaps the most simply formulated model showing avalanche behavior is the Bak-Sneppen model [1–7]: ''What could be simpler than replacing some random numbers with some other random numbers?'' [2]. Nevertheless, the exact solution of the Bak-Sneppen model is unknown even in one dimension. The value of the most fundamental quantity, i.e., of the upper critical dimension, is also still under discussion [8,9].

The formulation of the model is short indeed. A random number f_i from some distribution $\mathcal{P}(f)$ is placed at each site i of a lattice. One simultaneously replaces the smallest number f_{min} of these, and the random numbers at its nearest neighbor sites, by new random numbers from the distribution $\mathcal{P}(f)$, and afterwards the process is repeated.

Avalanches in the Bak-Sneppen model are defined in the following way. The f avalanche is defined as the sequence of steps at which f_{min} remains smaller than the given parameter f. (One may find a more detailed definition in Ref. [3].)

A very significant step toward understanding the nature of the avalanches in this model was made in Ref. [10] by Maslov, who introduced the so called avalanche hierarchy equation for the distribution $P(s,f)$ of f-avalanche sizes s (i.e., of temporal durations). From this *exact* equation, one may obtain additional relation between the critical exponents of the model. Unfortunately, an exact solution of the equation is known only for the mean field situation. Two first terms of the expansion from the mean field solution, i.e. from the higher critical dimension, were calculated in Ref. [11]. The precision of the results obtained by direct numerical integration of the avalanche hierarchy equation in its original form [10] is only comparable with the precision of the Monte Carlo simulations [3,12].

There is another way to obtain analytical results. It seems natural to start from the lower critical dimension, which is equal to zero for the Bak-Sneppen model, to find something similar to the well known $2+\epsilon$ expansion. Here from the avalanche hierarchy equation we derive some convenient relations which enable us find the singular relation between the critical exponents near zero dimension, and obtain values of the exponents at integer dimensions. Traditionally, one relates the exponents τ and μ [11,8] (see the definition of these exponents below). The total curve $\tau(\mu)$, with particular points for the integer dimensions, and the areas of applicability of the approaches of Ref. [11] and ourselves are depicted in Fig. 1.

For the distribution $\mathcal{P}(f)=e^{-f}$, $f>0$ [13], the avalanche hierarchy equation is of the form [10]

$$\frac{\partial P(s,f)}{\partial f}=\sum_{t=1}^{s-1} t^{\mu}P(t,f)P(s-t,f)-s^{\mu}P(s,f). \qquad (1)$$

Here, in the scaling region, s^{μ} gives the average number of distinct sites updated during an avalanche of the size s,

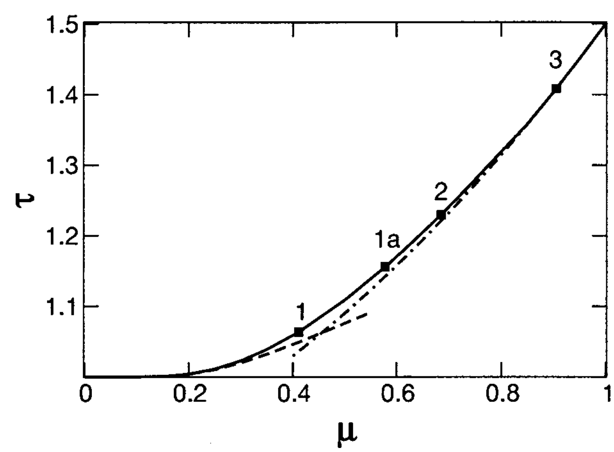

FIG. 1. The exponent τ vs μ calculated from Eq. (10) (also see Refs. [11,8]). The value of the exponent μ depends on the dimension d of the system. $\mu(d=0)=0$, $\mu=1$, at the upper critical dimension, i.e., at 4, as found in Ref. [9]. The dashed line is obtained from Eq. (17), i.e., by an expansion from the lower critical dimension; the dash-dotted line is the expansion [11] from the upper critical dimension. Points 1, 2, and 3 correspond to the Bak-Sneppen model in one, two, and three dimensions. Point 1a corresponds to the one-dimensional anisotropic Bak-Sneppen model.

*Electronic address: sdorogov@fc.up.pt
†Electronic address: jfmendes@fc.up.pt
‡Electronic address: ypogorel@fc.up.pt

where $\mu = d/D_f$, d is the dimension of the lattice, and D_f is the avalanche fractal dimension [3]. The physical meaning of the equation describing the hierarchical nature of avalanches in the Bak-Sneppen model is the following. The distribution $P(s,f)$ changes while f grows, for two reasons. First, two consecutive avalanches of size t and $s-t$ contribute to the avalanche of size s (the second avalanche starts from one of the sites changed during the first avalanche, which gives the factor t^{μ} in the sum). Second, some avalanches of size s merge into a larger avalanche.

For the problem under consideration, the exponent μ, which is equal to 0 at the lower critical dimension and equal to 1 at the upper critical dimension (simulations in [8] and [9] gave different values for it, $d_{uc}=8$ and 4, correspondingly), is the given parameter. All other exponents are related to μ by Eq. (1). Below the threshold f_c (i.e. in the symmetric phase) the solution of Eq. (1) has the following scaling form:

$$P(s,f) = s^{-\tau} F(s^{\sigma}(f_c - f)). \tag{2}$$

Equation (1) resembles nonlinear differential equations with a peaking regime [14]. For such equations, it is possible to find both exponents included in Eq. (2).

In Ref. [11], it was proposed to search for the Laplace transform of the distribution $P(s,f)$:

$$p(\alpha,f) = \sum_{s=1}^{\infty} P(s,f) e^{-\alpha s}. \tag{3}$$

Then Eq. (1) gives

$$\begin{aligned} -\frac{1}{1-p(\alpha,f)} \frac{\partial p(\alpha,f)}{\partial f} &= \sum_{s=1}^{\infty} P(s,f) s^{\mu} e^{-\alpha s} \\ &= (-1)^{\mu} \frac{\partial^{\mu} p(\alpha,f)}{\partial \alpha^{\mu}} \\ &= -\frac{1}{\Gamma(1-\mu)} \int_0^{\infty} dt\, t^{-\mu} \frac{\partial p(\alpha+t,f)}{\partial \alpha}, \end{aligned} \tag{4}$$

where $\partial^{\mu}/\partial \alpha^{\mu}$ denotes the fractional partial derivative (μ is certainly a noninteger), and the last expression is its integral representation. The scaling relation for the solution of Eq. (4) is

$$p(\alpha,f) = 1 - \alpha^{\tau-1} h\left(\frac{f_c - f}{\alpha^{\sigma}}\right). \tag{5}$$

Inserting Eq. (5) into Eq. (4), one obtains the usual relation between the critical exponents,

$$\tau = 1 + \mu - \sigma, \tag{6}$$

and the following integral-differential equation for the scaling function $h(x)$:

$$\Gamma(1-\mu) x \frac{h'(x)}{h(x)} = \int_0^x dy \frac{y h'(y) - \dfrac{\mu-\sigma}{\sigma} h(y)}{[1-(y/x)^{1/\sigma}]^{\mu}}. \tag{7}$$

Here $h'(x) \equiv dh(x)/dx$.

In Ref. [11], Eq. (7) was used to obtain the expansion from the mean field solution [5–7], but this seems to be inconvenient. Let us show that one may transfer it to a purely integral form. The following lines demonstrate how that integration may be done:

$$\begin{aligned} \Gamma(1-\mu) \frac{d \ln h(x)}{dx} &= \int_0^1 \frac{dz}{(1-z^{1/\sigma})^{\mu}} \\ &\quad \times \left[xz \frac{dh(xz)}{d(xz)} - \frac{\mu-\sigma}{\sigma} h(xz) \right] \\ &= \int_0^1 \frac{dz}{(1-z^{1/\sigma})^{\mu}} (xz)^{(\mu-\sigma)/\sigma+1} \\ &\quad \times \frac{d}{d(xz)} [(xz)^{-(\mu-\sigma)/\sigma} h(xz)] \\ &= x^{(\mu-\sigma)/\sigma+1} \int_0^1 \frac{dz}{(1-z^{1/\sigma})^{\mu}} z^{(\mu-\sigma)/\sigma+1} \\ &\quad \times \frac{1}{z} \frac{d}{dx} [x^{-(\mu-\sigma)/\sigma} z^{-(\mu-\sigma)/\sigma} h(xz)] \\ &= x^{(\mu-\sigma)/\sigma+1} \frac{d}{dx} x^{-(\mu-\sigma)/\sigma} \\ &\quad \times \int_0^1 dz \frac{h(xz)}{(1-z^{1/\sigma})^{\mu}}. \end{aligned} \tag{8}$$

Applying $\int_0^x dx$ to the first and last lines of Eq. (8), and then integrating by parts [one may choose $h(x=0)=1$ [11]], we obtain

$$\begin{aligned} \Gamma(1-\mu) \ln h(x) &= \int_0^1 \frac{dz}{(1-z^{1/\sigma})^{\mu}} \\ &\quad \times \left[x h(xz) - \left(\frac{\mu-\sigma}{\sigma} + 1\right) \int_0^x du\, h(uz) \right], \end{aligned} \tag{9}$$

and finally we obtain the equation for the scaling function $h(x)$ in the most convenient form

$$\begin{aligned} h(x) = \exp\Bigg\{ & \frac{1}{\Gamma(1-\mu)} \int_0^x \frac{dy}{[1-(y/x)^{1/\sigma}]^{\mu}} \\ & \times \left[h(y) - \frac{\mu}{\sigma} \frac{1}{y} \int_0^y dz\, h(z) \right] \Bigg\} \end{aligned} \tag{10}$$

[if one does not demand $h(0)=1$, $h(x)$ in the left parts of Eqs. (9) and (10) is $h(x)/h(0)$].

The asymptotic form of $h(x)$ for large x follows from the expansion of Eq. (3) in small α. Below the threshold, $h(x)$ has to be

$$h(x)\cong x^{(\mu-\sigma)/\sigma-1/\sigma}(c_0+c_1x^{-1/\sigma}+c_2x^{-2/\sigma}+\cdots). \tag{11}$$

This particular asymptotic behavior fixes the solution of Eq. (10) and the value of σ for any given μ. Substituting Eq. (11) into Eq. (7), Eq. (9), or Eq. (10), one obtains the sum rule

$$\int_0^\infty dx h(x)=\frac{1-(\mu-\sigma)}{\mu}\Gamma(1-\mu). \tag{12}$$

Note that if $h(x,\mu,\sigma)$ is a solution of Eq. (10), then $ch(cx,\mu,\sigma)$ is also a solution for any constant c.

Equations (10) and (12) are the set of equations that lead to the scaling function $h(x,\mu)$ and $\sigma(\mu)$. Instead of Eq. (12), one may equally well use the condition on the value of the exponent of the asymptote,

$$[xh'(x)/h(x)](x\to\infty)=\frac{(\mu-\sigma)-1}{\sigma}. \tag{13}$$

Hence the problem is reduced to the eigenvalue problem for the nonlinear equation [16].

Let us study the solution of the system for small μ. The expansion of the solution of Eq. (10) in x looks like

$$\begin{aligned}\ln h(x)=&\left(1-\frac{\mu}{\sigma}\right)B(\sigma,1-\mu)\left(\frac{\sigma x}{\Gamma(1-\mu)}\right)+\left(1-\frac{\mu}{\sigma}\right)\\ &\times B(\sigma,1-\mu)\left(1-\frac{1}{2}\frac{\mu}{\sigma}\right)B(2\sigma,1-\mu)\\ &\times\left(\frac{\sigma x}{\Gamma(1-\mu)}\right)^2+\left(1-\frac{\mu}{\sigma}\right)B(\sigma,1-\mu)\left[\frac{1}{2}\left(1-\frac{\mu}{\sigma}\right)\right.\\ &\left.\times B(\sigma,1-\mu)+\left(1-\frac{1}{2}\frac{\mu}{\sigma}\right)B(2\sigma,1-\mu)\right]\\ &\times\left(1-\frac{1}{3}\frac{\mu}{\sigma}\right)\\ &\times B(3\sigma,1-\mu)\left(\frac{\sigma x}{\Gamma(1-\mu)}\right)^3+\cdots\\ =&-\frac{\mu-\sigma}{\sigma}\sum_{n=1}^{\infty}\frac{1}{n}B(\sigma,1-\mu)\cdots B(n\sigma,1-\mu)\\ &\times\left(\frac{\sigma x}{\Gamma(1-\mu)}\right)^n+\cdots. \end{aligned} \tag{14}$$

We shall see that the quantity $(\mu-\sigma)/\sigma$ is the smallest parameter of the problem near the lower critical dimension. If one tends formally μ to 0, the last line of Eq. (14) tends to

$$\ln h(x)=-\frac{\mu-\sigma}{\sigma}\sum_{n=1}^{\infty}\frac{\Gamma(1+n\mu)}{nn!}\left(\frac{x}{\Gamma(1-\mu)}\right)^n, \tag{15}$$

and afterwards to

$$\ln h(x)=-\frac{\mu-\sigma}{\sigma}\sum_{n=1}^{\infty}\frac{1}{nn!}\left(\frac{x}{\Gamma(1-\mu)}\right)^n. \tag{16}$$

Thus, for small μ, the solution $h(x)$ behaves in the following way. For low enough x, the solution very slowly decreases from the value $h(0)=1$, and, in some crossover region $x\sim 1/\mu$, it reaches the asymptotic power tail [Eq. (11)]. We have to stress that three last limit equalities may be justified only for small x, and that the omitted terms in Eq. (14) do contribute to the power-law tail. Nevertheless, one may try to estimate $\sigma(\mu)$ for small μ by inserting Eqs. (15) and (16) into the sum rule (12). The solution of the first of the equations obtained in such a way is $\mu-\sigma=\exp\{-\mu^{-1}-2\gamma+O(\mu)\}$, where $\gamma=0.5772\ldots$ is Euler's constant, and the solution of the second one is of the form

$$\mu-\sigma=\tau-1=\exp\{-\mu^{-1}-\gamma+O(\mu)\}. \tag{17}$$

Thus the dependence is nonanalytical, but even the second term of the expansion cannot be defined by such estimation.

In fact we failed to obtain the value of the constant analytically. Nevertheless, Eqs. (10) and (12) are very convenient for numerics, since iterations of Eq. (10) converge. [One may start, for instance, from functions (15) or (16).] We checked the validity of relation (17) for small μ. The value of the constant in Eq. (17) obtained in such a way is 0.5771(5), i.e., it is indeed Euler's constant.

Solving Eq. (10) with the constraint (12) or (13), and the initial condition, one may easily obtain $\sigma(\mu)$ and $\tau(\mu)$ for any given μ. The values of the exponent μ are known from simulation at integer dimensions with much higher precision than the values of τ, because of the better available statistics [3,12]. Therefore, we can essentially improve the precision of the known value of τ. For the one-dimensional (1D) Bak-Sneppen model, we obtain $\tau=1.0637(5)+0.4(\mu-0.4114)$, where $\mu=0.4114(2)$ is the value obtained from the Monte Carlo simulation [12]. For the 2D Bak-Sneppen model, we obtain $\tau=1.229(1)+0.77(\mu-0.685)$, where $\mu=0.685(5)$ is the value obtained in Ref. [12]. (The last relations may be used to obtain better values of τ when more precise values of the exponent μ will be available.) Now the precision of τ coincides with that of μ. Note that these values are below the values of τ previously obtained from the simulation, $\tau(1\text{D})=1.073(3)$ and $\tau(2\text{D})=1.245(10)$ [12], but are in accordance with the less precise values found in Ref. [11] by direct numerical solution of the avalance hierarchy equation [Eq. (1)]. The value of the exponent τ of the 3D Bak-Sneppen model ($\mu=0.905$ [9]) may be obtained from an expansion from a higher critical dimension [11]. In Fig. 1, we show the curve $\tau(\mu)$ together with the points for the integer dimensions and the low-μ asymptote, [Eq. (17)], and the expansion from the upper critical dimension [11].

Of course, relation (17) is valid only for $\mu\ll 1$. Nevertheless, let us compare the value of τ at $\mu=0.4114$ obtained from Eq. (17), ($\tau=1.0494$), with the calculated above $\tau(1\text{D})$. One may see that these values are in qualitative agreement.

In the case of the 1D anisotropic Bak-Sneppen model (i.e., for the update of the extremal site and only one neighbor, for instance, the right), the exponents σ and μ are coupled by the additional relation $\sigma+\mu=1$ [15]. Hence one can find all the exponents of the problem. From Eq. (10) we obtained the value $\mu=0.5779(5)$. In Ref. [15], two different values of μ were obtained: $\mu=0.58$ from Eq. (1), and $\mu=0.588$ found in another way. The Monte Carlo simulation made in Refs. [17] and [18] gave $\mu=0.60(1)$ and $\mu=0.59(3)$, correspondingly. Therefore, we had to check our result. For that we numerically solved Eq. (7) with the initial condition $h(0)=1$ and constraint (12) or (13). The result is $\mu=0.5778(5)$. Thus the value $\mu=0.578$ seems to be more reliable, but the question is still open.

In summary, we have demonstrated that the simple transformation of the avalanche hierarchy equation made it convenient for analysis and numerics. We have obtained the nontrivial singular relation $\tau-1=\mu-\sigma=\exp\{-\mu^{-1}-\gamma+\cdots\}$, with Euler's constant γ, between the scaling exponents of the Bak-Sneppen model near zero dimension. Using the values of the exponent μ known from simulations, we have in fact, found all other exponents of the Bak-Sneppen model in one and two dimensions with the same high precision. We also obtained the exponents of the anisotropic 1D Bak-Sneppen model. Nevertheless, one should note that the main problem of obtaining the last independent critical exponent of the Bak-Sneppen model remains open.

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